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Advanced Linear Algebra (MA 409)
Problem Sheet - 17

Properties of Determinants

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1. Label the following statements as true or false.

   (a) If E is an elementary matrix, then $\det(E) = \pm 1$.

   (b) For any $A, B \in M_{n\times n}(F)$, $\det(AB) = \det(A) \cdot \det(B)$.

   (c) A matrix $M \in M_{n\times n}(F)$ is invertible if and only if $\det(M) = 0$.

   (d) A matrix $M \in M_{n\times n}(F)$ has rank n if and only if $\det(M) \neq 0$.

   (e) For any $A \in M_{n\times n}(F)$, $\det(A^t) = -\det(A)$.

   (f) The determinant of a square matrix can be evaluated by cofactor expansion along any column.

   (g) Every system of n linear equations in n unknowns can be solved by Cramer's rule.

   (h) Let $Ax = b$ be the matrix form of a system of n linear equations in n unknowns, where $x = (x_1, x_2, \ldots, x_n)^t$. If $\det(A) \neq 0$ and if M_k is the $n \times n$ matrix obtained from A by replacing row k of A by b^t, then the unique solution of $Ax = b$ is

$$x_k = \frac{\det(M_k)}{\det(A)} \quad \text{for } k = 1, 2, \ldots, n.$$

In Exercises 2-7, use Cramer's rule to solve the given system of linear equations.

2. $a_{11}x_1 + a_{12}x_2 = b_1$
   $a_{21}x_1 + a_{22}x_2 = b_2$
   where $a_{11}a_{22} - a_{12}a_{21} \neq 0$

3. $2x_1 + x_2 - 3x_3 = 5$
   $x_1 - 2x_2 + x_3 = 10$
   $3x_1 + 4x_2 - 2x_3 = 0$

4. $2x_1 + x_2 - 3x_3 = 1$
   $x_1 - 2x_2 + x_3 = 0$
   $3x_1 + 4x_2 - 2x_3 = -5$

5. $x_1 - x_2 + 4x_3 = -4$
   $-8x_1 + 3x_2 + x_3 = 8$
   $2x_1 - x_2 + x_3 = 0$

6. $x_1 - x_2 + 4x_3 = -2$
   $-8x_1 + 3x_2 + x_3 = 0$
   $2x_1 - x_2 + x_3 = 6$

7. $3x_1 + x_2 + x_3 = 4$
   $-2x_1 - x_2 = 12$
   $x_1 + 2x_2 + x_3 = -8$

8. Prove that an upper triangular $n \times n$ matrix is invertible if and only if all its diagonal entries are nonzero.

9. A matrix $M \in M_{n\times n}(\mathbb{C})$ is called **nilpotent** if, for some positive integer k, $M^k = O$, where O is the $n \times n$ zero matrix. Prove that if M is nilpotent, then $\det(M) = 0$.

10. A matrix $M \in M_{n\times n}(\mathbb{C})$ is called **skew-symmetric** if $M^t = -M$. Prove that if M is skew-symmetric and n is odd, then M is not invertible. What happens if n is even?

11. A matrix $Q \in M_{n\times n}(\mathbb{R})$ is called **orthogonal** if $QQ^t = I$. Prove that if Q is orthogonal, then $\det(Q) = \pm 1$.

12. For $M \in M_{n\times n}(\mathbb{C})$, let $\overline{M}$ be the matrix such that $(\overline{M})_{ij} = \overline{M_{ij}}$ for all i,j, where $\overline{M_{ij}}$ is the complex conjugate of M_{ij}.

    (a) Prove that $\det(\overline{M}) = \overline{\det(M)}$.

    (b) A matrix $Q \in M_{n\times n}(\mathbb{C})$ is called **unitary** if $QQ^* = I$, where $Q^* = \overline{Q^t}$. Prove that if Q is a unitary matrix, then $|\det(Q)| = 1$.

13. Let $\beta = \{u_1, u_2, \ldots, u_n\}$ be a subset of F^n containing n distinct vectors, and let B be the matrix in $M_{n\times n}(F)$ having u_j as column j. Prove that β is a basis for F^n if and only if $\det(B) \neq 0$.

14. Prove that if $A, B \in M_{n\times n}(F)$ are similar, then $\det(A) = \det(B)$.

15. Use determinants to prove that if $A, B \in M_{n\times n}(F)$ are such that $AB = I$, then A is invertible (and hence $B = A^{-1}$).

16. Let $A, B \in M_{n\times n}(F)$ be such that $AB = -BA$. Prove that if n is odd and F is not a field of characteristic two, then A or B is not invertible.

17. Prove that if A is an elementary matrix of type 2 or type 3, then $\det(AB) = \det(A)\cdot\det(B)$.

18. A matrix $A \in M_{n\times n}(F)$ is called **lower triangular** if $A_{ij} = 0$ for $1 \leq i < j \leq n$. Suppose that A is a lower triangular matrix. Describe $\det(A)$ in terms of the entries of A.

19. Suppose that $M \in M_{n\times n}(F)$ can be written in the form

$$M = \begin{pmatrix} A & B \\ O & I \end{pmatrix},$$

    where A is a square matrix. Prove that $\det(M) = \det(A)$.

20. Prove that if $M \in M_{n\times n}(F)$ can be written in the form

$$M = \begin{pmatrix} A & B \\ O & C \end{pmatrix},$$

    where A and C are square matrices, then $\det(M) = \det(A)\cdot\det(C)$.

21. Let $T : P_n(F) \to F^{n+1}$ be the linear transformation defined by $T(f) = (f(c_0), f(c_1), \ldots, f(c_n))$, where $c_0, c_1, \ldots, c_n$ are distinct scalars in an infinite field F. Let β be the standard ordered basis for $P_n(F)$ and γ be the standard ordered basis for F^{n+1}.

    (a) Show that $M = [T]_\beta^\gamma$ has the form

$$\begin{pmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^n \\ 1 & c_1 & c_1^2 & \cdots & c_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & c_n & c_n^2 & \cdots & c_n^n \end{pmatrix}.$$

    A matrix with this form is called a **Vandermonde matrix**.

(b) Prove that $\det(M) \neq 0$.

(c) Prove that

$$\det(M) = \prod_{0 \le i < j \le n} (c_j - c_i),$$

the product of all terms of the form $c_j - c_i$ for $0 \le i < j \le n$.

22. Let $A \in M_{n\times n}(F)$ be nonzero. For any m $(1 \le m \le n)$, an $m \times m$ **submatrix** is obtained by deleting any $n - m$ rows and any $n - m$ columns of A.

    (a) Let k $(1 \le k \le n)$ denote the largest integer such that some $k \times k$ submatrix has a nonzero determinant. Prove that $rank(A) = k$.

    (b) Conversely, suppose that $rank(A) = k$. Prove that there exists a $k \times k$ submatrix with a nonzero determinant.

23. Let $A \in M_{n\times n}(F)$ have the form

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & a_0 \\ -1 & 0 & 0 & \cdots & 0 & a_1 \\ 0 & -1 & 0 & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & a_{n-1} \end{pmatrix}.$$

    Compute $\det(A + tI)$, where I is the $n \times n$ identity matrix.

24. Let c_{jk} denote the cofactor of the row j, column k entry of the matrix $A \in M_{n\times n}(F)$.

    (a) Prove that if B is the matrix obtained from A by replacing column k by e_j, then $\det(B) = c_{jk}$.

    (b) Show that for $1 \le j \le n$, we have

$$A \begin{pmatrix} c_{j1} \\ c_{j2} \\ \vdots \\ c_{jn} \end{pmatrix} = \det(A) \cdot e_j.$$

    Hint: Apply Cramer's rule to $Ax = e_j$.

    (c) Deduce that if C is the $n \times n$ matrix such that $C_{ij} = c_{ji}$, then $AC = [\det(A)]I$.

    (d) Show that if $\det(A) \neq 0$, then $A^{-1} = [\det(A)]^{-1}C$.

The following definition is used in Exercises 26-27.

Definition. The **classical adjoint** of a square matrix A is the transpose of the matrix whose ij-entry is the ij-cofactor of A.

25. Find the classical adjoint of each of the following matrices.

a) $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$

b) $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

c) $\begin{pmatrix} -4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

d) $\begin{pmatrix} 3 & 6 & 7 \\ 0 & 4 & 8 \\ 0 & 0 & 5 \end{pmatrix}$

e) $\begin{pmatrix} 1-i & 0 & 0 \\ 4 & 3i & 0 \\ 2i & 1+4i & -1 \end{pmatrix}$

f) $\begin{pmatrix} 7 & 1 & 4 \\ 6 & -3 & 0 \\ -3 & 5 & -2 \end{pmatrix}$

g) $\begin{pmatrix} -1 & 2 & 5 \\ 8 & 0 & -3 \\ 4 & 6 & 1 \end{pmatrix}$

h) $\begin{pmatrix} 3 & 2+i & 0 \\ -1+i & 0 & i \\ 0 & 1 & 3-2i \end{pmatrix}$

26. Let C be the classical adjoint of $A \in M_{n\times n}(F)$. Prove the following statements.

   (a) $\det(C) = [\det(A)]^{n-1}$.

   (b) C^t is the classical adjoint of A^t.

   (c) If A is an invertible upper triangular matrix, then C and A^{-1} are both upper triangular matrices.

27. Let $y_1, y_2, \ldots, y_n$ be linearly independent functions in C^∞. For each $y \in C^\infty$, define $T(y) \in C^\infty$ by

$$[T(y)](t) = \det \begin{pmatrix} y(t) & y_1(t) & y_2(t) & \cdots & y_n(t) \\ y'(t) & y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & \vdots & & \vdots \\ y^{(n)}(t) & y_1^{(n)}(t) & y_2^{(n)}(t) & \cdots & y_n^{(n)}(t) \end{pmatrix}.$$

   The preceding determinant is called the **Wronskian** of $y, y_1, \ldots, y_n$.

   (a) Prove that $T : C^\infty \to C^\infty$ is a linear transformation.

   (b) Prove that $N(T)$ contains $span(\{y_1, y_2, \ldots, y_n\})$.

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